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FREE TOPOLOGICAL GROUPS AND DIMENSION[1]

BY

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ABSTRACT. For a completely regular space X we denote by $F(X)$ and $A(X)$ the free topological group of X and the free Abelian topological group of X, respectively, in the sense of Markov and Graev.

Let X and Y be locally compact metric spaces with either $A(X)$ topologically isomorphic to $A(Y)$ or $F(X)$ topologically isomorphic to $F(Y)$. We show that in either case X and Y have the same weak inductive dimension. To prove these results we use a Fundamental Lemma which deals with the structure of the topology of $F(X)$ and $A(X)$. We give other results on the topology of $F(X)$ and $A(X)$ and on the position of X in $F(X)$ and $A(X)$.

1. **Introduction.** The purpose of this paper is to continue the study of free topological groups begun by Markov [10] and Graev [6]. Graev showed that if X and Y are compact metric spaces such that $A(X)$ and $A(Y)$ are topologically isomorphic, then X and Y have the same dimension. In Theorem 2 we strengthen this result by showing that if X and Y are locally compact metric spaces such that $A(X)$ and $A(Y)$ are topologically isomorphic, then X and Y have the same weak inductive dimension. In Theorem 3 we give the analogous result for $F(X)$.

In order to prove these results on dimension we need a convenient way to study the topology which the free topology on $F(X)$ induces on Y. Let $F_n(X)$ be the set of all elements of $F(X)$ with length at most n. We define $A_n(X)$ similarly. We prove a Fundamental Lemma which says that if $x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$ is the reduced representation of a point of $F(X)$ where $x_i \in X$ and ϵ_i is 1 or -1 for $i = 1, 2, \ldots, n$, then a fundamental system of neighborhoods of $x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$ in $F_n(X)$ is formed by the family of all sets of the form $U_1^{\epsilon_1}U_2^{\epsilon_2}\cdots U_n^{\epsilon_n}$ where U_i is a neighborhood of x_i in X for $i = 1, 2, \ldots, n$. The analogous result is true for $A(X)$.

In addition to our Fundamental Lemma we need Theorem 1 in order to

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prove Theorems 2 and 3. Theorem 1 says that if X is any completely regular space and Y is a compact subset of $A(X)$ then $Y \subset A_k(X)$ for some positive integer k.

Another consequence of Theorem 1 is Theorem 4 which says that if X is not discrete then $F(X)$ and $A(X)$ are not locally compact. We also give an example of a paracompact space X such that $F(X)$ and $A(X)$ are not normal.

Finally let φ be a topological isomorphism of $F(X)$ onto itself. Then in Theorem 5 we show that $\varphi(X)$ algebraically generates $F(X)$ as its free topological group. An analogous result holds for $A(X)$.

2. **Preliminaries.** Before going into the main results of this paper we need some background material. Let X be a completely regular space. In this paper we use the definition of the free topological group $F(X)$ of X given by Graev [6]. We also adopt the notation used by Graev. Unless otherwise stated, all topological spaces and topological groups are Hausdorff and X will denote a completely regular space.

Let e be any point of X. The *free topological group $F(X)$ of the space X* is defined to be the unique topological group $F(X)$ which satisfies the following three properties.

(1) *$F(X)$ contains X as a subspace.*

(2) *There is no proper closed subgroup of $F(X)$ which contains X.*

(3) *If φ is a continuous mapping of X into a topological group G for which $\varphi(e)$ is the identity of G, then φ can be extended to a continuous homomorphism Φ of the topological group $F(X)$ into G.*

The definition of the *free Abelian topological group $A(X)$ of the space X* may be obtained by keeping (1) and (2) as they are and by considering only Abelian topological groups G in (3). The existence and uniqueness of $F(X)$ and $A(X)$ follow from Graev.

It also follows that X algebraically generates $F(X)$ and $A(X)$. From this we conclude that (2) may be replaced in the definitions of $F(X)$ and $A(X)$ by

(2′) *X generates $F(X)$ algebraically.*

In fact, Graev showed that the free topological group of X may be obtained by taking $X\backslash\{e\}$ as a set of generators for a free algebraic group G. Then we identify the point e of the space X with the identity element of the group G. If G is given the strongest topology which makes it into a topological group and also induces the original topology on X, then G becomes the free topological group of X. Similar remarks can be made about free Abelian topological groups.

We call the topology on $F(X)$ which makes $F(X)$ into a free topological group the *free topology* on $F(X)$ The *free topology* on $A(X)$ is defined similarly.

By the *length of an element* $\bar{x} \in F(X)$ we mean the smallest integer n such that $\bar{x} = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ where $x_i \in X$ and ϵ_i is 1 or -1 for $i = 1, 2, \ldots, n$. The *length of the word* $y_1^{\epsilon_1} y_2^{\epsilon_2} \cdots y_k^{\epsilon_k}$, where $y_i \in X$ and ϵ_i is 1 or -1 for $i = 1, 2, \ldots, k$, is k, even if the word can be reduced to something shorter.

Throughout this paper, unless otherwise stated, e will denote the identity of $F(X)$, and 0 the identity of $A(X)$. Unless otherwise stated, symbols like x_i and y_i will denote points in $F_1(X)$ or $A_1(X)$.

Suppose X is a metric space. Graev gave a procedure for extending a metric on X to a metric on $F(X)$ which is compatible with the group structure of $F(X)$ and whose topology is weaker than the free topology for $F(X)$. Suppose X is a completely regular space. Then the procedure will also extend a pseudo-metric on X to $F(X)$. If the pseudo-metric topology on X is weaker than the given topology on X, then the topology induced on $F(X)$ by the extended pseudo-metric is weaker than the free topology on $F(X)$. Similar remarks can be made for the group $A(X)$.

As the above mentioned procedure is basic to this paper we outline it for the reader. Let ρ be a pseudo-metric on the space X. For convenience we will denote the extension by ρ also. We first extend ρ to $F_1(X)$ by $\rho(x^{-1}, y^{-1}) = \rho(x, y)$ and $\rho(x, y^{-1}) = \rho(x^{-1}, y) = \rho(x, e) + \rho(e, y)$ for any points x and y in X.

Next we extend ρ to all of $F(X)$. Suppose $\bar{x}$ and $\bar{y}$ are two elements of $F(X)$. We define

$$\rho(\bar{x}, \bar{y}) = \inf\left\{\sum_{i=1}^{k} \rho(x_i', y_i') : x_1' x_2' \cdots x_k' = \bar{x} \text{ and } y_1' y_2' \cdots y_k' = \bar{y}\right\}.$$

The above infimum is taken over all representations $x_1' x_2' \cdots x_k'$ of $\bar{x}$ and $y_1' y_2' \cdots y_k'$ of $\bar{y}$ which have the same length. The points x_i' and y_i' are in $F_1(X)$ for $i = 1, 2, \ldots, k$. We allow k to vary.

Graev showed that the distance $\rho(\bar{x}, \bar{y})$ is actually achieved for some representations of $\bar{x}$ and $\bar{y}$. That is, $\rho(\bar{x}, \bar{y}) = \Sigma_{i=1}^{k} \rho(x_i', y_i')$ for some particular representations $x_1' x_2' \cdots x_k'$ of $\bar{x}$ and $y_1' y_2' \cdots y_k'$ of $\bar{y}$.

Let $\{\rho_\nu\}_{\nu \in I}$ be the family of all continuous pseudo-metrics on a completely regular space X. Extend ρ_ν to $F(X)$ for every $\nu \in I$. Let $\mathcal{T}$ be the least upper bound of the topologies on $F(X)$ determined by the ρ_ν for $\nu \in I$. Then $\mathcal{T}$ induces the original topology on X. Further, $\mathcal{T}$ makes $F(X)$ into a Hausdorff topological group. The free topology on $F(X)$ is stronger than $\mathcal{T}$ and, in general, the topologies are not the same.

3. **A fundamental lemma on the topology of $F(X)$.** This section is devoted to an important lemma which will be used later in the paper. Before giving this lemma we state another lemma which is a consequence of general topology.

LEMMA 1. *Let $x_1, x_2, \ldots, x_n$ be distinct points of a completely regular space X. Let U_i be a neighborhood of x_i in X for $i = 1, 2, \ldots, n$. Suppose that $U_i \cap U_j = \emptyset$ whenever $i \neq j$. Then there is a continuous pseudo-metric ρ on X such that $\rho(x, x_i) = 1$ whenever $x \notin U_i$ for $i = 1, 2, \ldots, n$.*

FUNDAMENTAL LEMMA. *Let X be a completely regular space. Let $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ be a point of $F_n(X)$ where $x_i \in X$, $x_i \neq e$, and ϵ_i is 1 or -1 for $i = 1, 2, \ldots, n$; and $x_j \neq x_{j+1}$ wherever $\epsilon_j = -\epsilon_{j+1}$ for $j = 1, 2, \ldots, n-1$. That is, the word $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ is in reduced form. Then a base for the neighborhood system of $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ in the subspace $F_n(X)$ is formed by the family of all sets of the form $U_1^{\epsilon_1}U_2^{\epsilon_2} \cdots U_n^{\epsilon_n}$ where U_i is a neighborhood of x_i in X for $i = 1, 2, \ldots, n$.*

PROOF. Let V be a neighborhood of $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ in $F(X)$ for the free topology. By continuity of multiplication we can find neighborhoods U_1', U_2', $\ldots, U_n'$ of $x_1^{\epsilon_1}, x_2^{\epsilon_2}, \ldots, x_n^{\epsilon_n}$ respectively in $F(X)$ such that $U_1' U_2' \cdots U_n' \subset V$. Define $U_i = U_i'^{\epsilon_i} \cap X$ for $i = 1, 2, \ldots, n$ where X is regarded as a subspace of $F(X)$. Then we have $U_1^{\epsilon_1}U_2^{\epsilon_2} \cdots U_n^{\epsilon_n} \subset V \cap F_n(X)$, and U_i is a neighborhood of x_i in X for $i = 1, 2, \ldots, n$. It follows that every neighborhood of $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ in the subspace $F_n(X)$ contains a set of the form $U_1^{\epsilon_1}U_2^{\epsilon_2} \cdots U_n^{\epsilon_n}$ where U_i is a neighborhood of x_i in X for $i = 1, 2, \ldots, n$.

Let U_i be a neighborhood of x_i in X for $i = 1, 2, \ldots, n$. We also require that the sets U_i satisfy conditions (i) $e \notin U_i$; (ii) $U_i \cap U_j = \emptyset$ whenever $x_i \neq x_j$; and (iii) $U_i = U_j$ whenever $x_i = x_j$ for all $i, j = 1, 2, \ldots, n$. It is sufficient to show that $U_1^{\epsilon_1}U_2^{\epsilon_2} \cdots U_n^{\epsilon_n}$ is a neighborhood of $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ in the subspace $F_n(X)$. Notice that every reduced word in $U_1^{\epsilon_1}U_2^{\epsilon_2} \cdots U_n^{\epsilon_n}$ has length exactly n.

By applying Lemma 1 we define a pseudo-metric ρ on X such that $\rho(x_i, x) = 1$ whenever $x \in X \backslash U_i$ for $i = 1, 2, \ldots, n$. We observe that $\rho(x_i, x_j) = 1$ whenever $x_i \neq x_j$, and $\rho(x_i, e) = 1$ for $i = 1, 2, \ldots, n$. We extend this pseudo-metric ρ to the entire free topological group $F(X)$ in the usual way, that is in the way which was outlined previously. We also call the extended pseudo-metric ρ, and recall that it induces a topology on $F(X)$ which is weaker than the free topology.

Let $U_i' \subset F(X)$ be defined for $i = 1, 2, \ldots, n$ by

$$U_i' = \{x \in F(X)\colon \rho(x, x_i) < 1/n\}.$$

We will show that

$$U_1'^{\epsilon_1}U_2'^{\epsilon_2} \cdots U_n'^{\epsilon_n} \cap F_n(x) \subset U_1^{\epsilon_1}U_2^{\epsilon_2} \cdots U_n^{\epsilon_n}$$

so that $U_1^{\epsilon_1}U_2^{\epsilon_2}\cdots U_n^{\epsilon_n}$ is a neighborhood of $x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$ in $F_n(X)$. Notice that $e\notin U_i'$ for $i=1,2,\ldots,n$.

Let $y_1y_2\cdots y_r$ and $y_1'y_2'\cdots y_r'$ be any two words in $F(X)$ of equal length r with y_i and y_i' in $F_1(X)$ for $i=1,2,\ldots,r$ but not necessarily in reduced form. We define

$$\varphi(y_1y_2\cdots y_r, y_1'y_2'\cdots y_r')=\sum_{i=1}^{r}\rho(y_i, y_i').$$

Notice that φ may take on different values for different representations of the same elements.

Let $w_i=y_{i1}y_{i2}\cdots y_{ik_i}$ be a point of $U_i'^{\epsilon_i}$ for $i=1,2,\ldots,n$. We assume further that $y_{i1}y_{i2}\cdots y_{ik_i}$ is a representation of w_i which achieves the distance to $x_i^{\epsilon_i}=x_{i1}x_{i2}\cdots x_{ik_i}$. That is,

$$\rho(x_i^{\epsilon_i}, w_i)=\sum_{j=1}^{k_i}\rho(x_{ij}, y_{ij})=\varphi(x_{i1}x_{i2}\cdots x_{ik_i}, y_{i1}y_{i2}\cdots y_{ik_i})$$

for $i=1,2,\ldots,n$ where the points x_{ij} and y_{ij} are in $F_1(X)$ for $i=1,2,\ldots,n$ and $j=1,2,\ldots,k_i$. Suppose further that

$$y_1\cdots y_{n'}=w_1\cdots w_n=y_{11}\cdots y_{1k_1}\cdots y_{n1}\cdots y_{nk_n}$$

is a point of $U_1'^{\epsilon_1}U_2'^{\epsilon_2}\cdots U_n'^{\epsilon_n}\cap F_n(X)$ where y_i is a point of $F_1(X)$ for $i=1,2,\ldots,n'$. Finally we assume $y_1y_2\cdots y_n'$ is in reduced form. We wish to show $n'=n$ and $y_i\in U_i^{\epsilon_i}$ for $i=1,2,\ldots,n$. Since $y_1y_2\cdots y_{n'}\in F_n(X)$ we know that $n'\leqslant n$.

We calculate

$$\begin{aligned}\rho(x_1^{\epsilon_1}\cdots x_n^{\epsilon_n}, y_1\cdots y_{n'})&=\rho(x_1^{\epsilon_1}\cdots x_n^{\epsilon_n}, w_1\cdots w_n)\\&\leqslant\varphi(x_{11}\cdots x_{1k_1}\cdots x_{n1}\cdots x_{nk_n}, y_{11}\cdots y_{1k_1}\cdots y_{n1}\cdots y_{nk_n})\\&=\varphi(x_{11}\cdots x_{1k_1}, y_{11}\cdots y_{1k_1})+\cdots+\varphi(x_{n1}\cdots x_{nk_n}, y_{n1}\cdots y_{nk_n})\\&=\rho(x_1^{\epsilon_1}, w_1)+\cdots+\rho(x_n^{\epsilon_n}, w_n)<\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}=1.\end{aligned}$$

Thus we have shown

$$\rho(x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}, y_1y_2\cdots y_{n'})<1. \tag{1}$$

Suppose $x_1'x_2'\cdots x_k'=x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}$ and $y_1'y_2'\cdots y_k'=y_1y_2\cdots y_{n'}$ are words with x_i' and y_i' in $F_1(X)$ for $i=1,2,\ldots,k$ which realize the distance $\rho(x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}, y_1y_2\cdots y_{n'})$. That is,

$$\rho(x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n}, y_1y_2\cdots y_{n'})=\sum_{i=1}^{k}\rho(x_i', y_i').$$

Consider the arrangement

$$x'_1x'_2 \cdots x'_k$$
$$(2) \qquad y'_1y'_2 \cdots y'_k.$$

Let us fix orders of cancellation for reducing the words $x'_1x'_2 \cdots x'_k$ and $y'_1y'_2 \cdots y'_k$ to their reduced forms $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ and $y_1y_2 \cdots y_{n'}$ respectively. Whenever we cancel elements in what follows we have these orders of cancellation in mind.

Suppose we consider an $x_i^{\epsilon_i}$ in the top line of (2) which remains when the top line is reduced to its reduced form. Under this $x_i^{\epsilon_i}$ is an element u_{i1}. If u_{i1} is an e or one of the y_j that remain when the bottom line is reduced to reduced form, then we stop. Otherwise u_{i1} is cancelled by a u_{i1}^{-1} occurring somewhere on the bottom line of (2). Above u_{i1}^{-1} is an element u_{i2} in the top line. If u_{i2} is either e or one of the $x_j^{\epsilon_j}$ that remain when the top line is reduced, then we stop. We continue this process until it stops as it must, since (2) is a finite arrangement. There are four possible cases.

Case 1. The process ends with an e in the bottom line of (2).

Case 2. The process ends with an e in the top line.

Case 3. The process ends with an element y_{j_i} on the bottom line which remains when the bottom line is reduced.

Case 4. The process ends with an element $x_{j_i}^{\epsilon_{j_i}}$ on the top line which remains when the top line is reduced.

We wish to show that with our particular words only Case 3 can occur. Then we will show that the process beginning with $x_i^{\epsilon_i}$ ends with y_i for $i = 1, 2, \ldots, n$. This will imply that $n = n'$.

The reader may show that neither of the first two cases can occur. Suppose Case 4 occurs for $x_i^{\epsilon_i}$. Then the total number of u_{ij} generated is odd, say $2m_i - 1$. The process thus ends with $x_{j_i}^{\epsilon_{j_i}}$ occurring over $u_{i,2m_i-1}$ in the arrangement (2). We calculate

$$\begin{aligned}
&\rho(x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}, y_1y_2 \cdots y_{n'}) \\
&\qquad \geqslant \rho(x_i^{\epsilon_i}, u_{i1}) + \rho(u_{i1}^{-1}, u_{i2}) + \rho(u_{i2}^{-1}, u_{i3}) + \cdots + \rho(u_{i,2m_i-1}^{-1}, x_{j_i}^{\epsilon_{j_i}}) \\
&\qquad = \rho(x_i^{\epsilon_i}, u_{i1}) + \rho(u_{i1}, u_{i2}^{-1}) + \cdots + \rho(u_{i,2m_i-1}, x_{j_i}^{-\epsilon_{j_i}}) \geqslant \rho(x_i^{\epsilon_i}, x_{j_i}^{-\epsilon_{j_i}}).
\end{aligned}$$

The way the pseudo-metric ρ was constructed assures us that $\rho(x_i^{\epsilon_i}, x_{j_i}^{-\epsilon_{j_i}}) \geqslant 1$ if $x_i^{\epsilon_i} \neq x_{j_i}^{-\epsilon_{j_i}}$. This contradicts (1). Thus the only way Case 4 can occur is for $x_i^{\epsilon_i}$ to equal $x_{j_i}^{-\epsilon_{j_i}}$. Since the word $x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ is in reduced form this implies $|i - j_i| \geqslant 2$. There is no loss of generality in assuming that $j_i \geqslant i + 2$. Thus we may assume that $x_i^{\epsilon_i}$ is to the left of $x_{j_i}^{\epsilon_{j_i}}$ in arrangement (2) and that $x_{i+1}^{\epsilon_{i+1}}$, and possibly more of the $x_j^{\epsilon_j}$, occurs between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$ in the top line of (2). We will show that Case 4 must also occur for the element $x_{i+1}^{\epsilon_{i+1}}$ and that $x_{j_{i+1}}^{\epsilon_{j_{i+1}}}$ is also between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$. This will lead to a contradiction which implies that Case 4 does not occur at all.

Before showing this we need a definition which is used both now and later in the proof. We say that an element of $F_1(X)$ in (2) is in *position p* or has *position number p* if it is situated in a position between the columns determined by u_{ip} and u_{ip}^{-1} in arrangement (2). Notice that an element may have many different position numbers or no position numbers at all. This may occur when the u_{ij} do not proceed strictly to the right but instead change directions one or more times while progressing from $x_i^{\epsilon_i}$ to $x_{j_i}^{\epsilon_{j_i}}$.

Notice also that if an element y lies in the bottom line of (2) and has odd position numbers $p_1, p_2, \ldots, p_t$, and possibly some even position numbers as well, then y must be cancelled by an element y^{-1} having exactly the same odd position numbers $p_1, p_2, \ldots, p_t$. This follows because when y has an odd position number p and y^{-1} is not also in this position, then y lies between u_{ip} and u_{ip}^{-1} on the bottom line and prevents their cancellation. The even position numbers may be different for such a pair y and y^{-1}. Similar remarks can be made about the even position numbers of elements occurring in the top line of (2).

Some of the elements u_{ip} and u_{ip}^{-1} may lie outside of the region between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$. Thus there may be points that lie to the left of $x_i^{\epsilon_i}$ or to the right of $x_{j_i}^{\epsilon_{j_i}}$ in (2) which have position numbers. However, if the sequence of terms u_{ij} and u_{ij}^{-1} does get outside of the region between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$, then it must return since the net sum of all its wanderings must be movement from $x_i^{\epsilon_i}$ to $x_{j_i}^{\epsilon_{j_i}}$. Thus we see that the total number of position numbers for a point which is not one of the terms u_{ij} or u_{ij}^{-1} and which lies outside of the region between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$ must be even. Similarly, the total number of position numbers of an element which is not one of the terms u_{ij} and u_{ij}^{-1} and which lies in the region between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$ is odd.

Consider the element $x_{i+1}^{\epsilon_{i+1}}$ which does not cancel when the top line is reduced. Since $i < i+1 < j_i$ we know that $x_{i+1}^{\epsilon_{i+1}}$ lies between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$ on the top line. Thus we may conclude that the total number of position numbers for $x_{i+1}^{\epsilon_{i+1}}$ is odd and that there are no even position numbers. Clearly $u_{i+1,1}$, which lies directly under $x_{i+1}^{\epsilon_{i+1}}$ has the same position numbers as $x_{i+1}^{\epsilon_{i+1}}$. Thus $u_{i+1,1}$ must cancel since it is an element in the bottom line with an odd, and hence positive, number of odd position numbers. It follows that Case 3 does not occur for $x_{i+1}^{\epsilon_{i+1}}$ with $u_{i+1,1}$ being $y_{j_{i+1}}$.

We shall study the sequence $u_{i+1,1}, u_{i+1,1}^{-1}, u_{i+1,2}, u_{i+1,2}^{-1}, \ldots$. Notice that if j is odd then $u_{i+1,j}$ and $u_{i+1,j}^{-1}$ are on the bottom line, while if j is even they are on the top line. Suppose that above $u_{i+1,j}^{-1}$ the element $x_{j_{i+1}}^{\epsilon_{j_{i+1}}}$ occurs. Then where convenient we will say that $u_{i+1,j+1}$ is equal to $x_{j_{i+1}}^{\epsilon_{j_{i+1}}}$. Similarly if the element below $u_{i+1,t}^{-1}$ is $y_{j_{i+1}}$, then we will say that $u_{i+1,t+1}$ is equal to $y_{j_{i+1}}$. We show that the following two properties are always satisfied.

(α) If $u_{i+1,j}$ is in the region between the positions of $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$ then $u_{i+1,j}$ has an odd number of odd position numbers and an even number of even position numbers.

(β) If $u_{i+1,j}$ is not in the region between the positions of $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$ then $u_{i+1,j}$ has an odd number of odd position numbers and also an odd number of even position numbers.

Before giving the proof we notice that whenever $u_{i+1,t}^{-1}$ occurs, the point $u_{i+1,t+1}$ occurs directly above or below it. Thus $u_{i+1,t}^{-1}$ and $u_{i+1,t+1}$ have the same position numbers.

The proof is by induction. We have shown the properties already for $u_{i+1,1}$ which has the same set of position numbers as $x_{i+1}^{\epsilon_{i+1}}$.

Assume the properties are true for $u_{i+1,t}$. We consider the following two cases.

Case (a). Suppose t is odd so that $u_{i+1,t}$ is on the bottom line. Then $u_{i+1,t}^{-1}$ must appear in (2), since by assumption there is an odd, and hence positive, number of odd position numbers. Thus the process cannot stop with $u_{i+1,t}$ being one of the elements y_j remaining when the bottom line is reduced. Since $u_{i+1,t}$ is on the bottom line we know $u_{i+1,t}^{-1}$ has exactly the same odd position numbers as $u_{i+1,t}$. Thus $u_{i+1,t}^{-1}$ and $u_{i+1,t+1}$ both have an odd number of odd position numbers. If $u_{i+1,t}^{-1}$ is between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$, then its total number of position numbers is odd so it and $u_{i+1,t+1}$ have an even number of even position numbers as desired. On the other hand, if $u_{i+1,t}^{-1}$ is not between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$, then the total number of position numbers is even so it and $u_{i+1,t+1}$ both have an odd number of even position numbers. In either event properties (α) and (β) are true for Case (a).

Case (b). Suppose t is even so that $u_{i+1,t}$ is on the top line. The proof for Case (b) is omitted.

It follows that properties (α) and (β) hold for every $u_{i+1,j}$ and $u_{i+1,j}^{-1}$. Suppose j is odd so that $u_{i+1,j}$ is on the bottom line of (2). Then by properties (α) and (β), $u_{i+1,j}$ must cancel since it must have an odd, and hence positive, number of odd position numbers. This implies Case 3 cannot occur for $x_{i+1}^{\epsilon_{i+1}}$. Suppose now that j is even and $u_{i+1,j}$ is outside of the region between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$. From property (β) we see that it must have an odd, and hence positive, number of even position numbers. Thus $u_{i+1,j}$ must cancel so that Case 4 cannot occur for $x_{i+1}^{\epsilon_{i+1}}$ with such a $u_{i+1,j}$ being $x_{j_{i+1}}^{\epsilon_{j_{i+1}}}$. But arrangement (2) is finite so the process must end somewhere. The only possibility is for it to end with $x_{j_{i+1}}^{\epsilon_{j_{i+1}}}$ in the region between $x_i^{\epsilon_i}$ and $x_{j_i}^{\epsilon_{j_i}}$.

We have shown that if Case 4 occurs for an element $x_i^{\epsilon_i}$ with $i < j_i$, then Case 4 also occurs for x_{i+1} and $i < j_{i+1} < j_i$. Evidently $i < i + 1 < j_{i+1} < j_i$ and $j_{i+1} \geqslant (i + 1) + 2 = i + 3$. By applying the same argument successively to $x_{i+1}^{\epsilon_{i+1}}, x_{i+2}^{\epsilon_{i+2}}, \ldots$ we find that Case 4 must occur for all the points $x_{i+1}^{\epsilon_{i+1}}$, $x_{i+2}^{\epsilon_{i+2}}, \ldots$ with $i + t$ being less than j_i for $t = 1, 2, \ldots$. This is impossible so Case 4 cannot occur.

Thus with our particular words in arrangement (2) only Case 3 can occur for any $x_i^{\epsilon_i}$, for $i = 1, 2, \ldots, n$, which is not cancelled when the top line is reduced. Since we know $n' \leqslant n$, this implies $n = n'$.

We still must show that $y_{j_i} = y_i$ for $i = 1, 2, \ldots, n$. To do this we will show that $j_i < j_{i+1}$ for $i = 1, 2, \ldots, n - 1$. Fix such an i. We know that $x_{i+1}^{\epsilon_{i+1}}$ is to the right of $x_i^{\epsilon_i}$ in (2).

Let us consider the sequence

$$u_{i1}, u_{i1}^{-1}, u_{i2}, u_{i2}^{-1}, \ldots, u_{i,2m_i}, u_{i,2m_i}^{-1}, u_{i,2m_i+1}$$

where $u_{i,2m_i+1} = y_{j_i}$. This sequence begins at $x_i^{\epsilon_i}$ and ends at y_{j_i}. Thus a point which is not a member of this sequence and lies in the region between $x_i^{\epsilon_i}$ and y_{j_i} must have an odd number of position numbers. Points outside of this region must have an even number of position numbers.

We will need the following two properties.

(α') If $u_{i+1\,j}$ is to the right of y_{j_i} then it has an even number of odd position numbers. If $u_{i+1\,j}$ is to the left of y_{j_i} then it has an odd number of odd position numbers.

(β') If $u_{i+1\,j}$ is to the right of $x_i^{\epsilon_i}$ then it has an even number of even position numbers. If $u_{i+1\,j}$ is to the left of $x_i^{\epsilon_i}$ then it has an odd number of even position numbers.

Properties (α') and (β') can be shown by induction by a proof similar to that used to prove (α) and (β). This proof is omitted.

Properties (α') and (β') hold for all points $u_{i+1\,j}, j = 1, 2, \ldots, 2m_{i+1}$ and also for the point $y_{j_{i+1}} = u_{i+1,2m_{i+1}+1}$. Now $y_{j_{i+1}}$ is on the bottom line, and it does not cancel when the bottom line is reduced. Thus it does not have any odd position numbers. By (α') this is impossible if it is to the left of y_{j_i}. Thus $y_{j_{i+1}}$ is to the right of y_{j_i} and $j_i < j_{i+1}$ for $i = 1, 2, \ldots, n - 1$ as desired. Since each of the integers j_t for $t = 1, 2, \ldots, n$ must be one of the numbers 1, 2, $\ldots$, n, it follows that $j_i = i$ for $i = 1, 2, \ldots, n$.

Thus we have shown that Case 3 applies to every $x_i^{\epsilon_i}$ which remains when the top line of (2) is reduced. Further we have show that $j_i = i$ for

$i = 1, 2, \ldots, n$. Going back to Case 3 we calculate for each $i = 1, 2, \ldots, n$,

$$\begin{aligned}\rho(x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}, y_1 y_2 \cdots y_n) & \\ &\geqslant \rho(x_i^{\epsilon_i}, u_{i1}) + \rho(u_{i1}^{-1}, u_{i2}) + \rho(u_{i2}^{-1}, u_{i3}) + \cdots + \rho(u_{i,2m_i}^{-1}, y_{j_i}) \\ &= \rho(x_i^{\epsilon_i}, u_{i1}) + \rho(u_{i1}, u_{i2}^{-1}) + \rho(u_{i2}^{-1}, u_{i3}) + \cdots + \rho(u_{i,2m_i}^{-1}, y_{j_i}) \\ &\geqslant \rho(x_i^{\epsilon_i}, y_{j_i}) = \rho(x_i^{\epsilon_i}, y_i).\end{aligned}$$

Thus by (1) we have $\rho(x_i^{\epsilon_i}, y_i) < 1$ for $i = 1, 2, \ldots, n$. Thus $y_i \in U_i^{\epsilon_i}$ for $i = 1, 2, \ldots, n$. That is,

$$U_1'^{\epsilon_1} U_2'^{\epsilon_2} \cdots U_n'^{\epsilon_n} \cap F_n(X) \subset U_1^{\epsilon_1} U_2^{\epsilon_2} \cdots U_n^{\epsilon_n}$$

as we set out to show. But $U_1'^{\epsilon_1} U_2'^{\epsilon_2} \cdots U_n'^{\epsilon_n} \cap F_n(X)$ is a neighborhood of $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ in $F_n(X)$ so the lemma follows.

A similar proof gives the Abelian case of the Fundamental Lemma.

The following corollary of the Fundamental Lemma is sometimes useful. This result has been announced by Arhangel'skiĭ [3].

COROLLARY 1. *Let X be a completely regular space. Then $F_n(X)$ and $A_n(X)$ are closed subsets of $F(X)$ and $A(X)$ respectively.*

PROOF. The corollary follows easily from the Fundamental Lemma. We give an independent proof. Let $\beta(X)$ be the Stone-Čech compactification of X. Then there exist a natural continuous monomorphism $\Phi\colon F(X) \longrightarrow F(\beta(X))$ such that $\Phi^{-1}(F_n(\beta(X))) = F_n(X)$. Since $\beta(X)$ is compact $F_n(\beta(X))$ is compact and hence closed in $F(\beta(X))$. Since Φ is continuous $F_n(X)$ is closed in $F(X)$.

4. **Determination of** $\dim X$ **by** $F(X)$ **or** $A(X)$. Before determining the dimension of X we need the following result.

THEOREM 1. *Let X be a completely regular space and let Y be a compact subset of $A(X)$. If the elements of Y are written in terms of the elements of X then the lengths of their reduced representations are bounded. That is Y is contained in $A_k(X)$ for some positive integer k.*

PROOF. Suppose Y is not contained in $A_k(X)$ for $k = 1, 2, \ldots$. Then we may choose a sequence $\{y_i\}_{i \geqslant 1}$ contained in Y such that the length of y_k in terms of X is at least $2^k + \Sigma_{i=1}^{k-1} n_i$ for $k = 1, 2, \ldots$, where n_i denotes the length of the reduced representation of y_i with respect to X.

Since Y is compact we know that the sequence $\{y_i\}_{i \geqslant 1}$ has at least one limit point $y \in Y$. Let $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n = y$ be the reduced representation of y in terms of elements x_i of X where ϵ_i is 1 or -1 for $i = 1, 2, \ldots, n$.

Similarly let $\epsilon_{i1}x_{i1} + \epsilon_{i2}x_{i2} + \cdots + \epsilon_{in_i}x_{in_i} = y_i$ be the reduced representation of y_i for $i = 1, 2, \ldots$, in terms of X where ϵ_{ij} is 1 or -1 for $j = 1, 2, \ldots, n_i$.

By adopting an idea used in the proof of Lemma 6.1 [6], we will define inductively continuous functions $\{f_i\}_{i\geq 1}$ from X into the additive real numbers. We define f_1 on X so that $f_1(0) = 0$ and $f_1(x_j) = 0$ for $j = 1, 2, \ldots, n$; $f_1(x_{1j}) = \epsilon_{1j}2^{-1}$ for $j = 1, 2, \ldots, n_1$ whenever $x_{1j} \neq x_i$ for $i = 1, 2, \ldots, n$; and $|f_1(x)| \leq 2^{-1}$ for all $x \in X$. Since X is completely regular, such a continuous function exists. Suppose the functions $f_1, f_2, \ldots, f_{k-1}$ have been chosen. Then define the function f_k on X so that $f_k(0) = 0$ and $f_k(x_j) = 0$ for $j = 1, 2, \ldots, n$; $f_k(x_{ij}) = 0$ for $i = 1, 2, \ldots, k-1$ and $j = 1, 2, \ldots, n_i$; $|f_k(x)| \leq 2^{-k}$ for all $x \in X$; and $f_k(x_{kj}) = \alpha_k\epsilon_{kj}2^{-k}$ whenever $x_{kj} \neq x_i$ for $i = 1, 2, \ldots, n$ and $x_{kj} \neq x_{it}$ for $i = 1, 2, \ldots, k-1$ and $t = 1, 2, \ldots, n_i$ and where α_k is 1 or -1 depending on whether $\Sigma_{i=1}^{k-1}\Sigma_{j=1}^{n_k}\epsilon_{kj}f_i(x_{kj})$ is nonnegative or negative.

Define the function f' from X into the additive group of real numbers by $f'(x) = \Sigma_{i=1}^{\infty} f_i(x)$ for all $x \in X$. The function f' is continuous on X. Extend f' to a continuous homomorphism f defined on all of $A(X)$ by

$$f(\epsilon_1' z_1 + \epsilon_2' z_2 + \cdots + \epsilon_t' z_t) = \sum_{i=1}^{t} \epsilon_i' f'(z_i)$$

where $z_i \in X$ and ϵ_i' is 1 or -1 for $i = 1, 2, \ldots, t$. We have for any $k = 1, 2, \ldots,$

$$|f(y_k)| = |f(\epsilon_{k1}x_{k1} + \epsilon_{k2}x_{k2} + \cdots + \epsilon_{kn_k}x_{kn_k})| = \left|\sum_{j=1}^{n_k} \epsilon_{kj}f'(x_{kj})\right|$$

$$= \left|\sum_{j=1}^{n_k}\sum_{i=1}^{\infty} \epsilon_{kj}f_i(x_{kj})\right| = \left|\sum_{j=1}^{n_k}\sum_{i=1}^{k} \epsilon_{kj}f_i(x_{kj})\right|$$

$$= \left|\sum_{i=1}^{k-1}\sum_{j=1}^{n_k} \epsilon_{kj}f_i(x_{kj}) + \sum_{j=1}^{n_k} \epsilon_{kj}f_k(x_{kj})\right|.$$

By using the definition of α_k we find that the two sums inside the last absolute value have the same sign. Thus we have

$$|f(y_k)| \geq \left|\sum_{j=1}^{n_k} \epsilon_{kj}f_k(x_{kj})\right|.$$

The way the functions f_k were chosen assures us that every nonzero term in the last sum has the same sign and magnitude 2^{-k}. Since $n_k \geq 2^k + \Sigma_{i=1}^{k-1} n_i$ we know that at least $n_k - (\Sigma_{i=1}^{k-1} n_i + n) \geq 2^k - n$ of these terms are not zero. Thus for any k large enough so that $2^k - n > 2^{k-1}$ or $2^{k-1} > n$ we have $|f(y_k)| > ½$. But

$$f(y) = f(\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n) = \sum_{i=1}^{n} \epsilon_i f'(x_i) = 0$$

so y cannot be a limit point of $\{y_k\}_{k\geq 1}$. This is a contradiction. The theorem follows.

Theorem 1 is not a new result. There is a much easier proof which establishes the theorem in both the Abelian and non-Abelian cases. Let X be a completely regular space and $\beta(X)$ its Stone-Čech compactification. Then there is a natural continuous monomorphism $\Phi: F(X) \longrightarrow F(\beta(X))$. Let Y be a compact subset of $F(X)$. Then $\Phi(Y)$ is a compact subset of $F(\beta(X))$. By Lemma 9.3 of Steenrod [19] and Theorem 4 of Graev [6] of $F(\beta(X))$ is contained in $F_n(\beta(X))$ for some n. Thus $Y \subset \Phi^{-1}(F_n(\beta(X))) = F_n(X)$.

The first proof is given because it does not use the fact that $A(X)$ is a free Abelian topological group on X. All that is used is that every continuous map from X into the additive group of real numbers R can be extended to a continuous homomorphism from $A(X)$ into R. This is a much weaker condition which is satisfied by $F(X, \underline{\underline{V}})$, the free topological group on X of $\underline{\underline{V}}$, where $\underline{\underline{V}}$ denotes any Abelian variety of topological groups containing R. For information on varieties of topological groups see [12], [13], and [14] by S. A. Morris.

Let us recall a few facts about dimension [15]. The empty set is said to have *weak inductive dimension* -1. A topological space X is said to have *weak inductive dimension* $\leqslant n$ if every point x in X has a fundamental system of neighborhoods whose boundaries have weak inductive dimension $\leqslant n-1$. We say the *weak inductive dimension* of X is equal to n if it is $\leqslant n$ but not $\leqslant n-1$. Weak inductive dimension is equivalent to the other standard definitions of dimension for a separable metric space.

In Graev [6] it is shown that if X and Y are compact metric spaces such that $A(X)$ and $A(Y)$ are topologically isomorphic then X and Y have the same dimension. By combining some ideas of Graev's proof with some of our own we are able to extend this result to the case where X and Y are locally compact metric spaces. The results also hold if $F(X)$ and $F(Y)$ are topologically isomorphic.

THEOREM 2. *Let X and Y be locally compact metric spaces with $A(X)$ topologically isomorphic to $A(Y)$. Then the spaces X and Y have the same weak inductive dimension.*

PROOF. We will regard Y as a subset of $A(X)$ which generates $A(X)$ as its free topological group. In this way we can write elements of X in terms of elements of Y and elements of Y in terms of elements of X.

Let $\eta_k(X)$ be the set of all points x in X where the dimension of X is at least k; that is, where $\dim_x X \geqslant k$. Suppose $\dim X \geqslant k$ so that $\eta_k(X)$ is not empty. Then from dimension theory [2, p. 42], we know for $x \in \eta_k(X)$ that $\dim_x \overline{\eta_k(X)} = \dim_x X \geqslant k$. Let X_0 be the nonempty compact closure in $\overline{\eta_k(X)}$ of an open set. Then X_0 is a compact subset of X which contains a dense set of points x such that $\dim_x X_0 \geqslant k$. It follows that the dimension of every open subset of X_0 is at least k.

Since X_0 is compact we know from Theorem 1 that the points of X_0 can be written as words of bounded length with respect to Y. Let n be the smallest positive integer for which $X_0 \subset A_n(Y)$. Let $x \in X_0$ have reduced representation $y_1 + y_2 + \cdots + y_n$ where $y_i \in A_1(Y)$ for $i = 1, 2, \ldots, n$ and where the set $S = \{(i, j)\colon y_i = y_j \text{ and } j \neq i\}$ contain a minimum number of pairs (i, j). Choose open sets $U_1, U_2, \ldots, U_n$ with compact closures in $A_1(Y)$ so that $0 \notin U_i$, either $U_i \subset Y$ on $U_i \subset Y^{-1}$, and $y_i \in U_i$ for $i = 1, 2, \ldots, n$; $U_i \cap U_j^{-1} = \emptyset$ for $i, j = 1, 2, \ldots, n$; $U_i \cap U_j = \emptyset$ whenever $(i, j) \notin S$; and $U_i = U_j$ whenever $(i, j) \in S$. Define

$$V = (U_1 + U_2 + \cdots + U_n) \cap X_0.$$

Then by the Fundamental Lemma, V is an open neighborhood of x in X_0 such that every element of V has length exactly n with respect to Y. Further, if $(y_1' + y_2' + \cdots + y_n') \in V$ then $y_i' = y_j'$ if and only if $(i, j) \in S$.

Every point $p \in V$ can be written uniquely in the form $p_1 + p_2 + \cdots + p_n$ where $p_i \in U_i$ for $i = 1, 2, \ldots, n$. Define functions f_i from V into U_i by $f_i(p) = p_i$ for $i = 1, 2, \ldots, n$ and for $p = p_1 + p_2 + \cdots + p_n \in V$. We will show that the functions f_i are continuous on V. Suppose $p = p_1 + p_2 + \cdots + p_n \in V$ and W_i is an open neighborhood of $f_i(p) = p_i$ in $A_1(Y)$. We may assume without loss of generality that $W_i \subset U_i$ for $i = 1, 2, \ldots, n$. We have

$$\begin{aligned} f^{-1}(W_i) &= \{p_1' + p_2' + \cdots + p_n' \in V\colon p_i' \in W_i\} \\ &= (U_1 + U_2 + \cdots + U_{i-1} + W_i + U_{i+1} + \cdots + U_n) \cap V. \end{aligned}$$

It follows that $f_i^{-1}(W_i)$ is a neighborhood of p in V so that f_i is continuous in V for $i = 1, 2, \ldots, n$.

First we consider the function f_1. Let $y_1' = x_{11} + x_{12} + \cdots + x_{1n_1}$ be a point of maximal length n_1 in $f_1(V)$ with respect to X. We also require that the set $S_1 = \{(i, j)\colon x_{1i} = x_{1j} \text{ and } i \neq j\}$ contain a minimum number of pairs (i, j). Choose open sets $V_{11}, V_{12}, \ldots, V_{1n_1}$ with compact closures in $A_1(X)$ so that $0 \notin V_{1i}$, $V_{1i} \subset X$ or $V_{1i} \subset X^{-1}$, and $x_{1i} \in V_{1i}$ for $i = 1, 2, \ldots, n$; $V_{1i} \cap V_{1j}^{-1} = \emptyset$ for $i, j = 1, 2, \ldots, n_1$; $V_{1i} \cap V_{1j} = \emptyset$ whenever $(i, j) \notin S_1$; $V_{1i} = V_{1j}$ whenever $(i, j) \in S_1$; and $(V_{11} + V_{12} + \cdots + V_{1n_1}) \cap A_1(Y) \subset U_1$.

Define W_1 to be an open set in $A(X)$ such that $W_1 \cap A_{n_1}(X) = V_{11} + V_{12} + \cdots + V_{1n_1}$ and $W_1 \cap A_1(Y) \subset U_1$. Define $U_1' = W_1 \cap A_1(Y)$. Then U_1' is an open neighborhood of y_1' in $A_1(Y)$ such that every point of $f_1(V) \cap U_1'$ has length exactly n_1 with respect to X and $U_1' \subset U_1$. Since $y_1' \in f_1(V)$, there is a point of the form $y_1' + y_{12} + y_{13} + \cdots + y_{1n} \in V \subset X_0$ with $y_{1i} \in U_i$ for $i = 2, 3, \ldots, n$. Define

$$V_1 = (U_1' + U_2 + U_3 + \cdots + U_n) \cap X_0.$$

Then $y_1' + y_{12} + y_{13} + \cdots + y_{1n} \in V_1 \subset V$; V_1 is open in X_0; and for any point $z_1 + z_2 + \cdots + z_n \in V_1$ with $z_i \in F_1(Y)$ for $i = 1, 2, \ldots, n$, we know z_1 has length exactly n_1 with respect to X. As $V_1 \subset V$, we know every element in V_1 has length exactly n with respect to Y.

We are now able to apply the same argument using f_2 instead of f_1 and V_1 instead of V. After this we continue by applying this argument successively to $f_3, f_4, \ldots, f_n$. At the end of this process we will have open sets V_{ij} with compact closures in $A_1(X)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$ so that: $0 \notin V_{ij}$ and either $V_{ij} \subset X$ or $V_{ij} \subset X^{-1}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$; $V_{i,j} \cap V_{i,t}^{-1} = \varnothing$ for $i = 1, 2, \ldots, n$ and $j, t = 1, 2, \ldots, n_i$; $V_{is} \cap V_{it} = \varnothing$ whenever $(s, t) \notin S_i$; $V_{is} = V_{it}$ whenever $(s, t) \in S_i$; and $(V_{i1} + V_{i2} + \cdots + V_{in_i}) \cap A_1(Y) \subset U_i$. During this process we let W_i be an open set in $A(X)$ such that $W_i \cap A_{n_i}(X) = V_{i1} + V_{i2} + \cdots + V_{in_i}$ and $W_i \cap A_1(Y) \subset U_i$ for $i = 1, 2, \ldots, n$. We also let $U_i' = W_i \cap A_1(Y)$ and $V_i = (U_1' + U_2' + \cdots + U_i' + U_{i+1} + \cdots + U_n) \cap X_0$ for $i = 1, 2, \ldots, n$. Each U_i' is a nonempty open subset of $A_1(Y)$ and each V_i is a nonempty open subset of $V \subset X_0$. We have $V \supset V_1 \supset V_2 \supset \cdots \supset V_n$.

It is V_n which interests us. Every point in V_n has length exactly n with respect to Y. If $z_1 + z_2 + \cdots + z_n$ is any point of V_n with each $z_i \in U_i' \subset A_1(Y)$ then z_i has length exactly n_i with respect to X for $i = 1, 2, \ldots, n$.

Let $z_{i1} + z_{i2} + \cdots + z_{in_i}$ be arbitrary points of $U_i' \cap f_i(V_n)$ for $i = 1, 2, \ldots, n$ where $z_{ij} \in V_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$. Define functions g_{ij} from $U_i' \cap f_i(V_n)$ into V_{ij} by $g_{ij}(z_{i1} + z_{i2} + \cdots + z_{in_i}) = z_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$. Since every point of $U_i' \cap f_i(V_n)$ has a unique representation of the form $z_{i1} + z_{i2} + \cdots + z_{in_i}$ these functions are well defined. As was the case for the functions f_i we know g_{ij} is continuous for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$.

We shall study the functions $g_{ij} \circ f_i$ from $V_n \subset X_0$ into $V_{ij} \subset X$. Suppose x is any point of V_n. Then we have the equation $x = y_1 + y_2 + \cdots + y_n = x_{11} + \cdots + x_{1n_1} + \cdots + x_{n1} + \cdots + x_{nn_n}$ for suitable choices of $y_i = x_{i1} + x_{i2} + \cdots + x_{in_i} \in U_i' \cap f_1(V_n)$, where $x_{ij} \in V_{ij}$ for $i = 1, 2, \ldots, n$ and

$j = 1, 2, \ldots, n_i$. From this equation we know the word

$$x_{11} + x_{12} + \cdots + x_{1n_1} + x_{21} + x_{22} + \cdots + x_{2n_2} + \cdots + x_{n1} + x_{n2} + \cdots + x_{nn_n}$$

reduces to simply x. It follows that for any $x \in V_n$ there are values of i and j for which $g_{ij} \circ f_i(x) = x_{ij} = x$. Let A_{ij} be the set of all fixed points of $g_{ij} \circ f_i$ in V_n. Then

$$V_n = \bigcup_{1 \leqslant i \leqslant n; 1 \leqslant j \leqslant n_i} A_{ij}.$$

In addition A_{ij} is closed in V_n for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n_i$. Since this is a finite union, we can choose integers s and t so that A_{st} contains a nonempty open set W_{st} in V_n. By taking a smaller set if necessary we assume $\overline{W}_{st} \subset V_n$ where the closure is taken in the metric space X_0.

We know $g_{st} \circ f_s$ is the identity on $\overline{W}_{st}$. Thus f_s is one-to-one on $\overline{W}_{st}$. We know f_s is continuous and $\overline{W}_{st}$ is compact. It follows that $f_s | \overline{W}_{sy}$ is a homeomorphism from $\overline{W}_{st}$ into $U_s' \subset F_1(Y)$. Recall that either $U_s' \subset Y$ or $U_s' \subset Y^{-1}$. Since Y and Y^{-1} are homeomorphic we know that in either case Y contains a copy $f_s(W_{st})$ of W_{st}. But W_{st} is a nonempty open subset of $X_0 = \overline{\eta_k(X)}$ so $\dim W_{st} \geqslant k$. It follows that $\dim Y \geqslant \dim f_s(W_{st}) \geqslant k$. Hence $\dim X \leqslant \dim Y$.

In the same way $\dim Y \leqslant \dim X$. Thus $\dim X = \dim Y$ as desired.

THEOREM 3. *Let X and Y be locally compact metric spaces with $F(X)$ topologically isomorphic to $F(Y)$. Then the spaces X and Y have the same weak inductive dimension.*

PROOF. By Markov [10], $A(X) = F(X)/\delta$ and $A(Y) = F(Y)/\delta$ where δ is the commutator subgroup. Thus $A(X) \cong A(Y)$. The result follows from Theorem 2.

5. **Miscellaneous results.** Next we take advantage of the theory we have developed to investigate what conditions the space X must satisfy in order for $F(X)$ and $A(X)$ to be locally compact, paracompact, or normal. If X has the discrete topology then $F(X)$ and $A(X)$ also have the discrete topology. Graev showed in C which follows his Lemma 6.1 that if X is not discrete then the topological groups $F(X)$ and $A(X)$ are not second countable. A close reading of his proof reveals that he shows that if X is not discrete then $F(X)$ and $A(X)$ are not first countable and hence not metrizable. The following consequence of Theorem 1 has also been shown by Dudley [4] and Abels [1].

THEOREM 4. *Let X be a completely regular space. If the topology of X is not discrete then the groups $F(X)$ and $A(X)$ are not locally compact.*

PROOF. Suppose U is a compact neighborhood of the identity element 0 in $A(X)$. By Theorem 1, $U \subset A_n(X)$ for some positive integer n. Let x_0 be a nonisolated point of X. Then nx_0 is a limit point of $\{nx: x \in X, x \neq x_0\}$. Thus 0 is a limit point of $\{nx - nx_0: x \in X, x \neq x_0\}$. Every point $nx - nx_0$ with $x \neq x_0$ has length $2n$ so U contains an element of length $2n$. This is impossible since $U \subset A_n(X)$. Thus there are no compact neighborhoods of 0 and $A(X)$ is not locally compact.

We know [6] that the mapping φ from $F(X)$ onto $A(X)$ defined by $\varphi(x_1x_2 \cdots x_n) = x_1 + x_2 + \cdots + x_n$ is both continuous and open. It follows that $F(X)$ is not locally compact either.

It is easy to see that if X is a completely regular space which is a countable union of compact spaces then the groups $F(X)$ and $A(X)$ are also countable unions of compact spaces. Thus $F(X)$ and $A(X)$ are paracompact. However if X is Lindelof and completely regular (i.e. paracompact) then $F(X)$ and $A(X)$ need not be paracompact. Let X be a completely regular space which is not normal. By constructing $F(X)$ and $A(X)$ for such a space, Markov [10] showed that a topological group or an Abelian topological group need not be normal. Our example will also show that the free topological group and free Abelian topological group of a normal space need not be normal.

EXAMPLE. Let X be the closed unit interval $[0, 1]$ with the half open interval topology. That is a base for the topology is formed by all half open intervals of the form $[x, y)$. X is a normal Lindelof space for which $F(X)$ and $A(X)$ are not normal. We leave the details of the proof to the reader. The proof uses the Fundamental Lemma and is very much like the standard proof used to show the topological product of X with itself is not normal. Additional relevant information can be found in [20].

Let G be a topological group. We denote by $\mathrm{Aut}(G)$ the set of all topological isomorphisms of G onto itself. Then $\mathrm{Aut}(G)$ is itself a group.

THEOREM 5. *Let X be a completely regular space. Let $\varphi \in \mathrm{Aut}(F(X))$. Then the subspace $\varphi(X)$ of $F(X)$ algebraically generates all of $F(X)$ as its free topological group.*

PROOF. There can be no nontrivial algebraic relations in $\varphi(X)$ as $\varphi^{-1} \in \mathrm{Aut}(F(X))$ and such relations are carried back into X by φ^{-1}. In addition $e \in \varphi(X)$. Thus $\varphi(X)$ algebraically generates a free group $\langle\varphi(X)\rangle \subset F(X)$. Let $x \in X$ with $\varphi^{-1}(x) = x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ where $x_i \in X$ and ϵ_i is 1 or -1 for $i = 1, 2, \ldots, n$. Then $x = \varphi(x_1^{\epsilon_1}x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}) = \varphi(x_1)^{\epsilon_1}\varphi(x_2)^{\epsilon_2} \cdots \varphi(x_n)^{\epsilon_n}$ so that $x \in \langle\varphi(X)\rangle$ for every $x \in X$. It follows that $\varphi(X)$ generates the entire free group $F(X)$.

Let us show that $F(X)$ is the free topological group of $\varphi(X)$. We verify the three properties given in the definition of a free topological group. First we observe that $\varphi(X) \subset F(X)$ and second we recall that $\varphi(X)$ algebraically generates $F(X)$. Now suppose h' is any continuous function from $\varphi(X)$ into a topological group G such that $h'(e)$ is the identity in G. We extend h' to a function h defined on all of $F(X)$ by $h(\varphi(x_1)^{\epsilon_1} \cdots \varphi(x_n)^{\epsilon_n}) = h'(\varphi(x_1))^{\epsilon_1} \cdots h'(\varphi(x_n))^{\epsilon_n}$ where $x_1, x_2, \ldots, x_n$ are points of X and ϵ_i is 1 or -1 for $i = 1, 2, \ldots, n$. Notice that since $\langle\varphi(X)\rangle = F(X)$ any word in $F(X)$ can be written in the form $\varphi(x_1)^{\epsilon_1}\varphi(x_2)^{\epsilon_2} \cdots \varphi(x_n)^{\epsilon_n}$.

The extended function h is clearly a homomorphism. We must show that it is continuous. Consider the continuous function $(h' \circ \varphi)|X$ from X into the group G. We know $F(X)$ is the free topological group of X so the extension g of $(h' \circ \varphi)|X$ to $F(X)$ defined by $g(x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}) = [(h \circ \varphi|X)(x_1)]^{\epsilon_1} \cdots [(h \circ \varphi|X)(x_n)]^{\epsilon_n}$ is continuous. A direct calculation shows that $h = g \circ \varphi^{-1}$. Thus h is continuous and $F(X)$ is the free topological group of $\varphi(X)$.

An analogous argument gives the Abelian case of this theorem.

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